Derived Equivalences of Azumaya Algebras on K3 Surfaces

Colin Ingalls and Madeeha Khalid

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Abstract

We consider moduli spaces of Azumaya algebras on K3 surfaces. In some cases we prove a derived equivalence between Azumaya algebras on on K3 surfaces. We also prove that when $\chi(A) = 0$, and $c_2(A)$ is within $2r$ of its minimal bound where r^2 is the rank of A, then the moduli space is a projective surface if it is non-empty.

1 Introduction

In this paper we investigate specific examples of moduli spaces of Azumaya algebras on K3 surfaces and their derived equivalences.

In recent years there has been considerable interest in derived equivalences between K3 surfaces. For instance in mathematical physics one is interested in field theories with a non trivial B-field. The B-field is a finite order element in the second cohomology. The question of equivalences between physical theories with B-fields, can be interpreted in terms of derived equivalences of twisted sheaves [21]. Twisted sheaves correspond to Azumaya algebras and so examples of derived equivalences between modules over Azumaya algerbas are also relevent to string theory.

Section 2 contains the preliminaries and basic definitions. In section 3 we review a classical example of an Azumaya algebra A on a K3 surface M, where M is a double cover of $\mathbb{P}^2$ branched over a sextic curve. We calculate invariants of A and prove that A has no deformations. In section 4 we construct a family of Azumaya algebras on M which are also in a division algebra. These Azumaya algebras have $\chi = 0$ and $c_2 = 8$. We prove that

the moduli space of these Azumaya algebras is a K3 surface X given by a triple intersection of quadric hypersurfaces in $\mathbb{P}^5$. This is like an inverse to the classical Mukai correspondence [19], in the context of Azumaya algebras.

In Section 5 we extend Bridgeland's [4] methods to the categories of modules over Azumaya algebras and prove some results on a derived equivalence between the two K3 surfaces X and M. Section 6 contains some results about Chern classes of Morita equivalent Azumaya algebras. In particular we show for a simple Azumaya algebra A of degree r, if $c_2(A)$ is within $2r$ of its lower bound then $c_2(A)$ is minimal. In some cases this implies that the moduli space of these Azumaya algebras is proper [1].

Our original motivation for studying this problem came from mathematical physics and a view towards formulating Mukai's results in the context of Azumaya algebras. We conclude with a discussion of these ideas.

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2 Brauer Group Basics

In this section we give some basic definitions and set up the notation for later use. The base field k is a algebraically closed field with characteristic zero. We assume that all our schemes are integral and reduced, unless specified otherwise. We also assume all cohomology is étale unless otherwise noted.

The Brauer group $Br(K)$, of a field K is the set of equivalence classes of central simple algebras over K. We have

$$Br(K) = H^2(Gal(\overline{K}/K), \overline{K}^*) = H^2_{et}(SpecK, \mathbb{G}_m).$$

This notion generalises to schemes also. For more details see [17].

2.1 Azumaya algebras

Definition 2.1 Let X be a reduced integral scheme. An *Azumaya algebra* A on X is a coherent sheaf of $\mathcal{O}_X$-algebras A such that

1. A is a locally free $\mathcal{O}_X$-module of rank n^2.

2. For every geometric point $p \in X$, the fibre $A_p := A \otimes \kappa(p)$ is isomorphic to $\kappa(p)^{n \times n}$, the algebra of $n \times n$ matrices over $\kappa(p)$.

The integer n is called the *degree* of A. The second condition is equivalent to A_p being a central simple $\kappa(p)$-algebra.

Etale locally an Azumaya algebra is the sheaf of endomorphisms of a vector bundle. More precisely, the following theorem is well known.

Theorem 2.2 *Let X be a scheme, and let A be a sheaf of $\mathcal{O}_X$-algebras, that is locally free of finite rank as a O_X-module. Then A is an Azumaya algebra on X if and only if for every point $p \in X$, there exists an étale neighbourhood $\pi : U \to X$ of p and a locally free sheaf $\mathcal{E}$ on U such that $\pi^* A \simeq \mathcal{E}nd_U(E)$.*

The following result is well known, for example see [1].

Proposition 2.3 *Let A be an Azumaya algebra. Then there is a trace pairing $tr : A \otimes A \to \mathcal{O}$ that is nondegenerate so $A \simeq A^*$ as $\mathcal{O}$-modules.*

Two Azumaya algebras A and A' are said to be *Morita equivalent* if and only if there exist locally free sheaves V, W such that

$$\mathcal{E}nd\ V \otimes A \ \simeq \ \mathcal{E}nd\ W \otimes A'.$$

This defines an equivalence relation on the set of Azumaya algebras on a scheme X, because $\mathcal{E}nd\ V \otimes \mathcal{E}nd\ W \ \simeq \ \mathcal{E}nd\,(V \otimes W)$. The tensor product of two Azumaya algebras is another Azumaya algebra, and this operation is compatible with the equivalence relation.

Definition 2.4 The Brauer group, Br(X) of a scheme X, is the group of Morita equivalence classes of Azumaya algebras, under the operation $[A][A'] = [A \otimes A']$. The identity element is $[\mathcal{O}_X]$ and $[A]^{-1} = [A^{\mathrm{op}}]$.

We recall the following facts.

Proposition 2.5 *Let K be a field of characteristic zero. Then there is a 1-1 correspondence between the sets*

$$\mathrm{Br}(K) \leftrightarrow \{\textit{division algebras } D \textit{ such that } Z(D) = K \textit{ and } \dim_K(D) < \infty\}/\simeq .$$

Definition 2.6 The *period* of an Azumaya algebra A, is its order in the Brauer group. The *index* is $\sqrt{\dim_K D}$ where K is the field of fractions of $\mathbb{Z}(A)$ and $A \otimes K \simeq D^{i \times i}$ with D a division algebra with centre K.

An Azumaya algebra is a division algebra if its period equals its index by [1]

Proposition 2.7 *Let X be a regular, integral, scheme and $k(X)$ the field of rational functions on X. There is an injection* $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$ *induced by restricting an Azumaya algebra to the generic point.*

Theorem 2.8 *Let A be an Azumaya algebra on X. Then any automorphism of A is Zariski locally an inner automorphism.*

For a proof, see [17].

The automorphism group of $GL_n(\mathcal{O}_X)$ is $PGL_n(\mathcal{O}_X)$, so An Azumaya algebra A corresponds to an element of $H^1(X, PGL_n(\mathcal{O}_X))$. We denote the image of A in $H^1(X, PGL_n(\mathcal{O}_X))$ by $c(A)$. If $A = \mathcal{E}nd\ V$ for some locally free sheaf V then $c(A)$ is trivial in $H^1(X, PGL_n(\mathcal{O}_X))$.

We have a short exact sequence

$$1 \to \mathcal{O}^* \to GL_n(\mathcal{O}_X) \to PGL_n(\mathcal{O}_X) \to 1$$

In the associated long exact sequence, we have

$$H^1(X, \mathcal{O}^*) \to H^1(X, GL_n(\mathcal{O}_X)) \to H^1(X, PGL_n(\mathcal{O}_X)) \xrightarrow{\delta} H^2(X, \mathcal{O}^*);$$

The maps δ are compatible for varying n, and

$$\delta(c(A) \otimes c(A')) = d(c(A) \otimes d(c(A'))).$$

So we get a natural injection $Br(X) \hookrightarrow H^2(X, \mathcal{O}^*)$.

In many cases this is an isomorphism. In particular when X is a smooth projective variety.

Definition 2.9 Let A be an Azumaya algebra of degree n on X. The *Brauer-Severi variety*, $BS(A)$, of an Azumaya algebra A, is the scheme parameterising rank n left ideals of A.

It is isomorphic to a $\mathbb{P}^{n-1}$ bundle over X. The class of $BS(A)$ in $H^1(X, \mathrm{PGL}_n)$ is the same as the class of A.

We mention some results on moduli of Azumaya algebras from [1] which we use later. We use the same notation as in [1]. Let X be a smooth projective surface and let $\alpha \in Br(X)$ be a Brauer class. If S is a scheme, we

consider Azumaya algebras A_S of degree n over $X_S = X \times S$ of the following type: $(*)$ A_S has degree n and for every geometric point s of S, the Brauer class of $A \otimes \kappa(s)$ is the one induced from α.

Let $\overline{\mathcal{A}}$ denote the functor, where $\overline{\mathcal{A}}(S)$ is the set of isomorphism classes of Azumaya algebras satisfying $(*)$ above. If A_S is a family of Azumaya algebras, the Chern class $c_2(A_S) \otimes \kappa(p)$ for $p \in S$ is a locally constant function on S. Let $\overline{\mathcal{A}}_c$ denote the functor such that $\overline{\mathcal{A}}_c(S)$ is the set of Azumaya algebras A_s such that $c_2(A_S) \otimes \kappa(p)) = c$ for $p \in S$. We have the following theorem due to Artin and deJong [1].

Theorem 2.10 *[Theorems 8.7.6, 8.7.7] Let $\overline{\mathcal{A}}_c$ be as above. Then we have the following*

1. *If $H^1(X, \mu_n) = 0$ then the functor $\overline{\mathcal{A}}_c$ is representable as an algebraic space.*

2. *If c denotes the minimal Chern class for the Brauer class α, then the algebraic space $\overline{\mathcal{A}}_c$ if non empty is proper.*

Recall that a proper algebraic space of dimension two is a projective surface.

We work with Azumaya algebras A over an algebraic K3 surface X.

Definition 2.11 A K3 surface X is a smooth, simply connected, complex surface with trivial canonical bundle.

The cohomology and Hodge decomposition of a K3 surface is completely determined [2].

For a K3 surface X we have

$$Br(X) = H^2_{et}(X, \mathcal{O}^*) = H^2_{an}(X, \mathcal{O}^*)_{tors} \simeq {T_X}^* \otimes \mathbb{Q}/\mathbb{Z}.$$

There are several references for this, see for instance [10].

We say that A is *simple* if $H^0(X, A) = k$.

Proposition 2.12 *Let X be a regular, integral scheme and E a vector bundle on X. Let A be an Azumaya algebra on X with $r^2 = \operatorname{rk} A$.*

1. *If E is a simple vector bundle then $A = \mathcal{E}nd\ E$ is simple.*

2. *If X is K3 and A is simple then $H^2(X, A) = k$*

3. *If X is K3 and A is simple, then $c_2(A) \geq 2r^2 - 2$.*

Proof. (1) This is just by definition, since E is simple means $\operatorname{End} E = k$, so $h^0(A) = h^0(\mathcal{E}nd\ E) = 1$.
(2) By Serre Duality we get $H^0(A) = H^2(A^*)$. But $A \simeq A^*$ via the trace pairing $A \otimes A \to \mathcal{O}$, so $H^2(X, A) = k$.
(3) The Riemann-Roch theorem implies $\chi(A) = h^0(A) - h^1(A) + h^2(A) = [\operatorname{ch} A.\operatorname{td} \mathcal{T}_X]_2$. Since $A \simeq A^*$, we have $c_1(A) = 0$, so $\operatorname{ch} A = r^2 - c_2 t^2$ and $\operatorname{td} \mathcal{T}_X = 1 + 2t^2$. So $\chi(A) = 2r^2 - c_2 \geq h^0(A) + h^0(A^*)$ hence $2r^2 - c_2 \geq 2$. □

3 The Koszul–Clifford Example

In this section we review a classical construction of an Azumaya algebra. We calculate its invariants and prove that it has no deformations.

3.1 Quadrics

For the reader's convenience we review some facts about quadratic forms on vector spaces. Let V denote an n dimensional complex vector space. By a *quadratic form* Q on V we mean a symmetric bilinear pairing

$$Q : V \times V \to k.$$

If we choose a basis $k^n \simeq V$ then There is an $n \times n$ symmetric matrix $S = (s_{ij})$ corresponding to Q such that for any $v, w \in V$,

$$Q(v, w) = vSw^T$$

In terms of coordinates on V, we have that $Q(v, v)$ is a homogeneous quadratic polynomial and hence defines an associated *quadric hypersurface*

$$\overline{Q} = \{[v] \in \mathbb{P}V : Q(v, v) = 0\}$$

in the projective space $\mathbb{P}V$. Conversely any quadric hypersurface $\overline{Q}$ in $\mathbb{P}V$, defined by the zero locus of a homogeneous polynomial of degree 2 lifts to a quadratic form Q.

We define the *rank* of Q to be the rank of the matrix S. If rank of S is n or equivalently if $\overline{Q}$ is smooth then Q is called *nondegenerate* otherwise Q is called *degenerate.*

A linear subspace $W \subset V$ is an *isotropic subspace* for Q if $Q_{|W} \equiv 0$ so $Q(u, v) = 0$ for all $u, v \in W$, or equivalently $\mathbb{P}W \subset \overline{Q}$. If the dimension of V is $2n$ and dimension of W is n then W is said to be *maximal isotropic.* The *orthogonal Grassmanian*

$$OGr(n, 2n, Q) = \{W \subset V : \dim W = n, Q_{|W} \equiv 0\}$$

is the variety of maximal isotropic subspaces of Q. Equivalently we can consider the quadric hypersurface $\overline{Q} \subset \mathbb{P}V$ in which case the the maximal isotropic subspaces correspond to families of $\mathbb{P}^{n-1}$ contained in $\overline{Q}$. If Q has rank $2n$, then $OGr(n, 2n, Q)$ is smooth of dimension $n(n-1)/2$ and consists of two disjoint connected components. If Q has rank $2n-1$ then $OGr(n, 2n, Q)$ is still smooth, of dimension $n(n-1)/2$ but now has only one component ([9] Chapter 6).

The Clifford algebra $\mathrm{Cl}(Q)$ of a quadratic form Q on a vector space V is defined as

$$\mathrm{Cl}(Q) := k\langle V\rangle / v \otimes v = Q(v, v), \text{ where } k\langle V\rangle = \bigoplus V^{\otimes n}.$$

The even degree terms form a subalgebra knows as the *even Clifford algebra* $\mathrm{Cl}_0(Q)$. In the next subsection we extend these notions to vector bundles.

3.2 A Classical Construction

Let Q_0, Q_1, Q_2 be rank 6 quadratic forms on k^6 and $\overline{Q}_0, \overline{Q}_1, \overline{Q}_2$ the associated quadric hypersurfaces in $\mathbb{P}^5$. Let $\overline{\mathcal{Q}}$ be the net of quadrics spanned by $\overline{Q}_0, \overline{Q}_1$ and $\overline{Q}_2$. Their intersection X is the base locus of $\overline{\mathcal{Q}}$ and is in general a smooth K3 surface of degree 8 in $\mathbb{P}^5$. We can also view $\mathcal{Q}$ as a quadratic form on the trivial vector bundle $\mathcal{O}^6_{\mathbb{P}^2}$, with values in $\mathcal{O}_{\mathbb{P}^2}(1)$

$$\mathcal{Q} : \mathrm{Sym}^2(\mathcal{O}^6_{\mathbb{P}^2}) \to \mathcal{O}_{\mathbb{P}^2}(1).$$

We denote by $\overline{\mathcal{Q}}$ the vanishing locus $V(\mathcal{Q})$ in the corresponding projective bundle $\mathbb{P}^2 \times \mathbb{P}^5$. For generic choice of Q_0, Q_1, Q_2, the degenerate quadrics have rank 5 and $C = V(\det(\mathcal{Q}))$ is a smooth degree 6 curve in $\mathbb{P}^2$. We assume that this is the case unless specified otherwise.

More generally we can define a quadratic form on a vector bundle with values in a line bundle.

Definition 3.1 Let V be a rank n vector bundle on a variety Z. By a quadratic form $\mathcal{Q}$ on V with values in a line bundle $\mathcal{L}$, we mean a map

$$\mathcal{Q} : \mathrm{Sym}^2 V \to \mathcal{L}.$$

The rank of $\mathcal{Q}$ restricted to the generic fibre is defined to be the *rank* of $\mathcal{Q}$. The fibrewise vanishing locus $V(\mathcal{Q})$ in $\mathbb{P}V$, defines a family of quadric hypersurfaces over the base variety Z. We call this the *quadric bundle* $\overline{\mathcal{Q}}$ over the base Z.

In fact we can define other algebraic structures associated to $(\mathcal{Q}, V)$. If $\mathcal{L} = \mathcal{O}$, there is a well known construction of the Clifford Algebra

$$\mathrm{Cl}(\mathcal{Q}) = T(V)/(v \otimes v - \mathcal{Q}(v, v))$$

where $v \in \mathcal{O}(V)$, [3]. The relations defining $\mathrm{Cl}(Q)$ are in $\mathrm{Sym}^\bullet V$. Locally we have

$$\mathrm{Cl}(\mathcal{Q}) = \mathcal{O} \oplus V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^n V.$$

The even subalgebra is

$$\mathrm{Cl}_0(\mathcal{Q}) = \mathcal{O} \oplus \wedge^2 V \oplus \wedge^4 V \oplus \cdots .$$

If $\mathcal{L} \neq \mathcal{O}$, there isn't a notion of a Clifford algebra as such. However we can still form an algebra with a sheaf structure

$$\mathrm{Cl}_0(Q) = \mathcal{O} \oplus (\wedge^2 V \otimes \mathcal{L}^{-1}) \oplus (\wedge^4 V \otimes \mathcal{L}^{-2}) \oplus \cdots$$

We basically follow the construction in [3].

Take the tensor product $T(V, \mathcal{L})$ of the $\mathbb{Z}$ graded algebras $\oplus V^{\otimes i}$ and $\oplus \mathcal{L}^i$,

$$T(V, \mathcal{L}) := \bigoplus V^{\otimes i} \otimes_Z \oplus \mathcal{L}^{-j}.$$

$\mathrm{Sym}^2(V) \subset V \otimes_Z V$ so we may consider $\mathcal{Q}$ as a relation in $T(V, \mathcal{L})$ and define the Clifford algebra $\mathrm{Cl}(\mathcal{Q})$ to be the quotient of $T(V, \mathcal{L})$ with defining relations $\mathcal{Q}$. More precisely let $I \subset T(V, \mathcal{L})$ be the two sided ideal generated by sections of the form $t \otimes t - \mathcal{Q}(t)$, for all $t \in \mathcal{O}(V)$. Then $\mathrm{Cl}(Q) := T(V, \mathcal{L})/I$. $\mathrm{Cl}(\mathcal{Q})$ is no longer bigraded but if we assign degree 1 to V and degree 2 to $\mathcal{L}^{-1}$, then the relation $t \otimes t - \mathcal{Q}(t)$ is homogeneous of degree 2. The degree 0 part $\mathrm{Cl}_0(\mathcal{Q})$ is called the *even Clifford Algebra* associated to the quadratic form $\mathcal{Q}$ on V. If $\mathcal{L} = \mathcal{O}$ we get the usual even Clifford algebra.

Definition 3.2 Let $\mathcal{Q} : \mathrm{Sym}^2 V \to \mathcal{L}$ be a quadratic form on a vector bundle of rank n. let $T(V, \mathcal{L})$ be the tensor algebra

$$T(V, \mathcal{L}) := \bigoplus V^{\otimes i} \otimes_Z \oplus \mathcal{L}^{-j}.$$

Assign degree 1 to V and degree 2 to $\mathcal{L}^{-1}$. Then the *even Clifford algebra* $\mathrm{Cl}_0(\mathcal{Q})$ of the quadric bundle $\mathcal{Q}$ is defined as the degree 0 subalgebra of

$$T(V, \mathcal{L})/(t \otimes t - \mathcal{Q}(t), \text{ for all } t \in \mathcal{O}(V)).$$

For our next result we need the following notions. Let A and B be $\mathbb{Z}_2$ graded algebras. We define a new graded algebra $A \otimes B$ where the multiplication has the sign conventions $(a \otimes b)(a' \otimes b') = (-1)^{\deg(b)\deg(a')} aa' \otimes bb'$.

We have the easy Lemma [14];

Lemma 3.3 *Let B be a $\mathbb{Z}_2$ graded algebra. Then there exists an isomorphism of graded algebras $\phi : M_2(k) \otimes B \to M_2(B)$.*

Let H be the quadratic form given by the hyperbolic lattice

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is well known that $\mathrm{Cl}(H) \simeq M_2(k)$ and $\mathrm{Cl}_0(H) \simeq k \oplus k$. It follows that $\mathrm{Cl}(H) \otimes B \simeq M_2(B)$. In particular

$$M_2(B)_0 = \begin{bmatrix} B_0 & B_1 \\ B_1 & B_0 \end{bmatrix}.$$

Proposition 3.4 *Let $\mathcal{Q} : \mathrm{Sym}^2 \mathcal{O}^6_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)$ be a quadratic form on the trivial bundle $\mathcal{O}^6_{\mathbb{P}^2}$. Let $\phi : M \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along the curve $V(det\mathcal{Q})$. Then $\mathrm{Cl}_0(\mathcal{Q}) = \phi_* A$, where A is an Azumaya algebra over M.*

Proof. This follows from local calculations of the Clifford algebra, by generalising the methods in [14] to modules over $k[[t]]$. Since the quadratic form has rank six over any point p not in $V(\det \mathcal{Q})$ we compute that $\mathrm{Cl}_0(\mathcal{Q}) \otimes k(p) \simeq M_4(k)^{\oplus 2}$. Next we need to compute the behaviour over a local transverse slice of the sextic. We may assume that $\mathcal{Q} = H \oplus H \oplus q_t$ where

$$q_t = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}.$$

We first consider the quadratic form given by $Q_t = H \oplus q_t$.

$$\mathrm{Cl}(Q_t) = \mathrm{Cl}(H) \otimes \mathrm{Cl}(q_t) \simeq M_2(k) \otimes \mathrm{Cl}(q_t)$$

$$\mathrm{Cl}_0(Q_t) = \begin{bmatrix} \mathrm{Cl}_0(q_t) & \mathrm{Cl}_1(q_t) \\ \mathrm{Cl}_1(q_t) & \mathrm{Cl}_0(q_t) \end{bmatrix}.$$

Let e_1, e_2, e_3, e_4 denote the standard basis for the underlying $k[[t]]$ module. Then $\mathrm{Cl}_1(q_t) = k\langle e_3, e_4\rangle/(e_3^2 - 1, e_4^2 - t, e_3e_4 + e_3e_4)$ and $\mathrm{Cl}_0(q_t) \simeq k[[\sqrt{t}]]$. The element e_3 in $\mathrm{Cl}_1(q_t)$ is invertible. An easy computation shows that $e_3 \mathrm{Cl}_1(q_t) = \mathrm{Cl}_1(q_t)e_3 = \mathrm{Cl}_0(q_t)$. Moreover $e_3 \mathrm{Cl}_0(q_t)e_3 = \mathrm{Cl}_0(q_t)$ is an inner automorphism. Now conjugation by the element

$$\begin{bmatrix} 1 & 0 \\ 0 & e_3 \end{bmatrix}$$

gives $\mathrm{Cl}_0(Q_t) \simeq M_2(\mathrm{Cl}_0(q_t)) \simeq M_2(k[[\sqrt{t}]])$. The generalisation to $\mathcal{Q}$ is essentially the same argument.

So $\mathrm{Cl}_0(\mathcal{Q})$ is an Azumaya algebra over its centre, whose underlying scheme M is a double cover of $\mathbb{P}^2$ branched over the locus of degenerate quadrics. □

Given a K3 surface M which is a double cover of a plane, branched over a smooth curve C, there are $2^9(2^{10}+1)$ inequivalent nets of quadrics $\mathcal{Q}$ such that $V(\det(\mathcal{Q})) = C$. So we get $2^9(2^{10}+1)$ non isomorphic Azumaya algebras on M which have the same invariants as A above. We describe this correspondence in more detail in section 4.

3.3 Numerical invariants

We use Koszul algebras to calculate invariants of A. There are several equivalent ways of defining a Koszul algebra. We use the following:

Definition 3.5 Consider a minimal free resolution of k

$$\to R^{\oplus n_2} \xrightarrow{d_2} R^{\oplus n_1} \xrightarrow{d_1} R \xrightarrow{d} k \to 0.$$

Then R is Koszul if and only if the entries of d_i are all of degree one.

The Koszul dual of an algebra is defined as the Yoneda algebra $R^! := \mathrm{Ext}^\bullet(k,k)$. We use the following results from [16].

Proposition 3.6 *The homogeneous coordinate ring of a complete intersection of quadrics is Koszul.*

Proposition 3.7 *If R is Koszul, then $H_R(t)H_{R^!}(-t) = 1$ where $H_R(t)$ is the Hilbert series of R.*

When $R = k\langle V\rangle/J$ for $J \subset V \otimes V$, the Koszul dual has the nice form

$$R^! = k\langle V^*\rangle/J^\perp$$

For a proof, see [16].

Consider the family $\mathcal{Q}$ of quadratic forms on $V \simeq k^6$ as in the previous section. The base locus X of $\overline{\mathcal{Q}}$ in $\mathbb{P}^5$ is a degree 8 K3 surface and is a complete intersection of three quadrics $\overline{Q}_0, \overline{Q}_1, \overline{Q}_2$. Let $y = (y_0, y_1, y_2, y_3, y_4, y_5)$ denote the coordinates on V, and let f_0, f_1, f_2 be the respective defining polynomials of the quadric hypersurfaces.

The homogeneous coordinate ring R, of X is Koszul and is given by

$$R = \mathbb{C}\langle V\rangle/J, \text{ where } J := \{[y_i, y_j], f_0(y, y), f_1(y, y), f_2(y, y)\}$$

The Hilbert series of X is

$$H_R(t) = \frac{(1-t^2)^3}{(1-t)^6} = \frac{(1+t)^3}{(1-t)^3}$$

Since $H_R(t) = \dfrac{1}{H_{R^!}(-t)}$, it follows from Proposition 3.7 that $H_R(t) = H_{R^!}(t)$. We compute this Hilbert series and get

$$\frac{(1+t)^3}{(1-t)^3} = -1 + \sum_i (4i^2 + 2)t^i$$

We also have the equality of algebras

$$v_2(R^!) := \bigoplus_{n\geq 0} R^!_{2n} = \bigoplus_{n\geq 0} H^0(M, A(n)),$$

as in [5, 13]. So we see that we may interpret $R^!$ as a polarization of A by the invertible bimodule $A(1/2)$, but it will be more convenient to use the polarization $\mathcal{O}_M(1)$ which we denote by h.

The Hilbert Series of $v_2(R^!)$ consists of the even terms of $H_{R^!}(t)$, so

$$H_{v_2(R^!)} = -1 + \sum_n (16n^2 + 2)t^n = 1 + 18t + 66t^2 + \cdots$$

By the Hirzebruch-Riemann-Roch formula we get $\chi(A(n)) = 16n^2 + 32 - c_2(A)$.

Since $\mathcal{O}_M(1)$ is ample we know that $\chi(A(n)) = h^0(A(n))$ for $n >> 0$. Comparing coefficients with the known Hilbert series $H_{v_2(R^!)}(t)$, we get $\chi(A(n)) = 16n^2 + 2$ for all n.

In particular, $\chi(A) = h^0(A) - h^1(A) + h^2(A) = 2$. From the first term of the Hilbert series it follows that $h^0(A) = 1$, hence $h^2(A) = 1$ and $h^1(A) = 0$. Recall from Section 2, that for a simple Azumaya algebra A, $c_2(A) \geq 2r^2 - 2$. In our case, $\mathrm{rk}(A) = 16$, and $c_2(A) = 30$, so this Azumaya algebra obtains its possible lower bound for $c_2(A)$.

To prove our next result we need the following result [1].

Proposition 3.8 *The first order deformations of an Azumaya algebra, A, fixing its centre M, are given by $H^1(M, A/\mathcal{O}_M)$ with obstructions in $H^2(M, A/\mathcal{O}_M)$.*

Proposition 3.9 *Let A be as in Proposition 3.4. Then A has no deformations.*

Proof. There is a short exact sequence

$$0 \to \mathcal{O}_M \to A \to A/\mathcal{O}_M \to 0$$

which is split by the trace map $tr : A \to \mathcal{O}_M$. From the associated long exact sequence of cohomology it follows that $H^1(M, A/\mathcal{O}_M) = H^2(M, A/\mathcal{O}_M) = 0$. Therefore we conclude that A has no deformations from the lemma above. $\square$

Recall from Section 2, Definition 2.6 that the period of an Azumaya algebra A is its order in the Brauer group while its degree is given by $\sqrt{\dim_{Z(\mathcal{A})} A}$. Also an Azumaya algebra is in a division algebra if its degree equals its period. In some cases we can determine these invariants. For instance we have the following [8] 1.5.3.

Theorem 3.10 *Let A be a Clifford algebra of a quadratic form. Then the period of $k(A)$ is two or one. If $k(A)$ is a division algebra then its period is two.*

So we see that A has period 2 and degree 4, and so is certainly not in a division algebra. There is however an interesting reduction of A which allows us to obtain an Azumaya algebras B which have index 2 and period 2 and are Morita equivalent to A.

4 Moduli of Azumaya algebras

In this section we construct a family of Azumaya algebras on M, where $\phi : M \to \mathbb{P}^2$ is a double cover of $\mathbb{P}^2$ branched along a sextic curve C. These algebras have index 2 and period 2 and so are contained in a division algebra. We first describe how to associate to M a net of quadrics $\overline{\mathcal{Q}} \subset \mathbb{P}^5$.

4.1 Theta characteristics and nets of quadrics

Definition 4.1 A theta characteristic on a smooth curve C is a line bundle L such that $L^{\otimes 2} \cong \omega_C$. It is called even if $H^0(L) = 0 \pmod 2$ and odd otherwise.

Given an ineffective theta characteristic we get a family of symmetric matrices $\mathcal{S}$ defining a quadric bundle $\mathcal{Q}$ with $V(\det \mathcal{Q}) = C$. We state this in the following theorem [22].

Theorem 4.2 *Let $\phi : M \to \mathbb{P}^2$ be a double cover of $\mathbb{P}^2$ branched along a smooth sextic curve C. Let L be an even theta characteristic on C such that $H^0(L) = 0$.*

1. *Then we have a resolution of L;*

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^6 \xrightarrow{\mathcal{S}} \mathcal{O}_{\mathbb{P}^2}(-1)^6 \to L \to 0$$

   where $\mathcal{S}$ is a symmetric matrix of linear forms on $\mathbb{P}^2$. The matrix $\mathcal{S}$ defines a quadric bundle $\mathcal{Q}$ and $V(\det \mathcal{Q}) = C$. The base locus X of the corresponding net $\overline{\mathcal{Q}}$ in $\mathbb{P}^5$ is smooth.

2. *Conversely given such an $\mathcal{S}$, its cokernel is a theta characteristic on C with $H^0(L) = 0$.*

There are $2^{g-1}(2^g+1)$ inequivalent even theta characteristics on a curve of genus g. The curve C has genus 10, and for generic C the theta characteristics

are not effective so we get $2^9(2^{10}+1)$ different nets of quadrics for each M. Since C is smooth, the base locus X of $\overline{\mathcal{Q}}$ in $\mathbb{P}^5$ is also smooth ([22] Lemmas 1.1 and 2.6). Recall that X is a K3 surface of degree 8 in $\mathbb{P}^5$ realised as a complete intersection of three quadric hypersurfaces. In general $\mathrm{Pic}(X)$ and $\mathrm{Pic}(M)$ have rank 1.

If $h^0(L)=2$, $\mathrm{Pic}(M)$ has rank 2. Then M is isomorphic to X and embeds in $\mathbb{P}^5$ as a degree 8 surface. However it is not a complete intersection. We describe this case in more detail in [12].

4.2 Reduction

There is a construction that allows one to pass from a quadric bundle of dimension d to a quadric bundle of dimension $d-2$, and depends on choosing a section. Let M, X and $\mathcal{Q}$ be as in Theorem 4.2. We pick a point $x \in X$. Note that x is a smooth point of every quadric in the net (Lemma 2.6 [22]). Let $a \in \mathbb{P}^2$, let $\overline{Q}_a$ be the corresponding quadric hypersurface in $\mathbb{P}^5$ and let $T_x\overline{Q}_a$ be the projective tangent space to $\overline{Q}_a$ at x. Then $\overline{Q}_a \cap T_x\overline{Q}_a$ is a singular quadric hypersurface in $T_x\overline{Q}_a$ consisting of the union of all lines in $\overline{Q}_a$ passing through x.

Let $F \cong \mathbb{P}^3$ be a subspace of $T_x\overline{Q}_a$ not containing x. If a is not in C then $\overline{Q}_a$ is smooth, and $T_x\overline{Q}_a \cap \overline{Q}_a$ is a cone over a smooth quadric surface $\overline{Q}_x(a)$ in F. If $a \in C$ then $\overline{Q}_a$ is singular with a vertex and $T_x\overline{Q}_a \cap \overline{Q}_a$ is a cone over a singular quadric surface $\overline{Q}_x(a)$ with a vertex. So given any a, projecting away from x, we get a quadric surface $\overline{Q}_x(a)$, which is singular or smooth depending on whether a is in C or not. Varying the a and projecting away from x defines a family of quadric surfaces in a $\mathbb{P}^3$ bundle. In fact this family of quadric surfaces is the vanishing locus of a quadratic form on a vector bundle.

Proposition 4.3 *Let $\phi : M \to \mathbb{P}^2$ be a double cover of $\mathbb{P}^2$ branched along a smooth sextic curve C and let X and $\mathcal{Q}$ be as in Theorem 4.2 for some choice of a theta characteristic L where $H^0(L)=0$. Note that $C = V(\det \mathcal{Q})$. Let $x \in X$. Then projection away from x induces a quadratic form $\mathcal{Q}_x$ on the bundle $V \simeq \Omega^1_{\mathbb{P}^2}(1) \oplus \mathcal{O}^2_{\mathbb{P}^2}$.*

$$\mathcal{Q}^x : \mathrm{Sym}^2 V \to \mathcal{O}_{\mathbb{P}^2}(1)$$

The quadratic form $\mathcal{Q}_x$ has rank 4. The degenerate quadrics have rank 3 and are paramterised by C.

Proof.

Let $a = [a_0; a_1, a_2]$ denote homogeneous coordinates on $\mathbb{P}^2$. We may assume with out loss of generality that $x = [0:0:0:0:0:1]$. Let $\mathcal{S}$ be the family of symmetric matrices corresponding to $\mathcal{Q}$ and let $T_x\mathcal{Q}$ be the rank 5 subbundle of $\mathcal{O}^6_{\mathbb{P}^2}$, given by

$$0 \to T_x\mathcal{Q} \to \mathcal{O}^6_{\mathbb{P}^2} \xrightarrow{\langle x\mathcal{S}, \cdot\rangle} \mathcal{O}_{\mathbb{P}^2}(1) \to 0$$

Projection away from x induces a map $\pi : \mathcal{O}^6_{\mathbb{P}^2} \to \mathcal{O}^5_{\mathbb{P}^2}$. We get a commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & T_x\mathcal{Q} \arrow[r] \arrow[d] & \mathcal{O}^6_{\mathbb{P}^2} \arrow[r, "<x\mathcal{S}, \cdot>"] \arrow[d, "\pi"] & \mathcal{O}_{\mathbb{P}^2}(1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V \arrow[r] & \mathcal{O}^5_{\mathbb{P}^2} \arrow[r, "\langle \pi(x\mathcal{S}), \cdot\rangle"] & \mathcal{O}_{\mathbb{P}^2}(1) \arrow[r] & 0
\end{tikzcd}$$

Here $\langle \pi(x\mathcal{S}), \cdot\rangle$ denotes the induced map from $\mathcal{O}^5$ to $\mathcal{O}_{\mathbb{P}^2}(1)$, after projection. After an appropriate change of basis of $\mathcal{O}^5_{\mathbb{P}^2}$ we may assume that $\langle \pi(x\mathcal{S}), \cdot\rangle = \langle (a_0, a_1, a_2, 0, 0), \cdot\rangle$. Recall the Euler sequence on $\mathbb{P}^2$;

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^3 \to T_{\mathbb{P}^2} \to 0$$

Dualising and tensoring with $\mathcal{O}_{\mathbb{P}^2}(1)$ we get;

$$0 \to \Omega^1_{\mathbb{P}^2}(1) \to \mathcal{O}^3_{\mathbb{P}^2} \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}^2}(1) \to 0,$$

where for $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in H^0(\mathcal{O}^3_{\mathbb{P}^2})$, $\sigma(\lambda) = a_0\lambda_0 + a_1\lambda_1 + a_2\lambda_2$. Then $V = \ker\sigma \oplus \mathcal{O}^2_{\mathbb{P}^2}$, i.e. $V \simeq \Omega^1_{\mathbb{P}^2}(1) \oplus \mathcal{O}^2_{\mathbb{P}^2}$.

The degenerate quadrics in $\mathcal{Q}_x$ are exactly those obtained from $\mathcal{Q}$ via projection away from x so they are parameterised by C and have rank 3. $\square$

The even Clifford algebras of these quadric bundles are rank 4 Azumaya algebras on M.

Theorem 4.4 *Let M, X and $\mathcal{Q}_x : \operatorname{Sym}^2 V \to \mathcal{O}_{\mathbb{P}^2}(1)$ be as in Proposition 4.3. Then $\mathrm{Cl}_0(\mathcal{Q}_x) = \phi_* B_x$, where B_x is a rank 4 Azumaya algebra on M with $\chi(B_x) = 0$ and $c_2(B_x) = 8$.*

Proof. We pick a point x in X and reduce $\mathcal{Q}$ to a quadratic form $\mathcal{Q}_x$ on V where $V \simeq \Omega^1_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$. The degenerate quadrics are paramterised by

C. Now we use the same arguments as in Proposition 3.4. Given a point p away from C, $\mathcal{Q}_{x|_p}$ is non degenerate of rank 4 so $\mathrm{Cl}_0(\mathcal{Q}_x) \otimes \kappa(p) \simeq M_2(k)^{\oplus 2}$. Locally across a transversal slice of the sextic $\mathrm{Cl}_0(\mathcal{Q}_x) = M_2([[k\sqrt{t}]])$. So the even Clifford algebra $\mathrm{Cl}_0(\mathcal{Q}_x)$ is an Azumaya algebra B_x on M, so $\phi_* B_x = \mathrm{Cl}_0(\mathcal{Q}_x)$. As a sheaf $\mathrm{Cl}_0(\mathcal{Q}_x) \simeq \mathcal{O} \oplus (\wedge^2 V \otimes \mathcal{O}(-1)) \oplus (\wedge^4 V \otimes \mathcal{O}(-2))$. It follows that $H^0(B_x) = k$. By a computation using the Grothendieck-Riemann-Roch formula we show that $\chi(B_x) = 0$ and hence $c_2(B_x) = 8$. $\square$

The Azumaya algebras B_x have rank 4. They have index 2 and period 2 and hence are in a division algebra. Suppose we have two points x and y in X. The Azumaya algebras B_x and B_y are isomorphic if and only if their associated Brauer-Severi varieties $BS(B_x), BS(B_y)$ resp. are isomorphic. We now prove the following theorem.

Theorem 4.5 *Let $\mathcal{Q} : \mathrm{Sym}^2 V :\to \mathcal{L}$, be a quadratic form on a rank 4 vector bundle V, with values in $\mathcal{L}$, and let $\mathrm{Cl}_0(\mathcal{Q})$ be the associated even Clifford algebra. Suppose that the rank of the degenerate quadrics is 3. Then the Brauer-Severi Variety $BS\,\mathrm{Cl}_0(\mathcal{Q})$ of $\mathrm{Cl}_0(\mathcal{Q})$ is isomorphic to the orthogonal Grassmanian $OGr(2,4,\mathcal{Q})$ of $\mathcal{Q}$.*

Proof. We define a map from the variety of isotropic planes in V to the set of left ideals in $\mathrm{Cl}_0(\mathcal{Q})$. Let W be a local section of isotropic planes in V. There is a natural map from $\wedge^2 W$ to $\mathrm{Cl}_0(\mathcal{Q})$. Let f denote a generator for the image of $\wedge^2 W$ in $\mathrm{Cl}_0(\mathcal{Q})$. Then $\mathrm{Cl}_0(\mathcal{Q}).f$ is in ideal in $\mathrm{Cl}_0(\mathcal{Q})$. Let $(y : z : w : x)$ denote local co ordinates on V. Locally across a transversal slice of the sextic $\mathcal{Q}$ is given by

$$Q_t = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Then we get two families of isotropic planes: $W_{\sqrt{t}} = \{(y, z, ay + a\sqrt{t}z, \frac{1}{a}(y - \sqrt{t}z), a \in k\}$ and $W_{-\sqrt{t}} = \{(y, z, ay - a\sqrt{t}z, \frac{1}{a}(y + \sqrt{t}z), a \in k\}$. So we get a family of isotropic planes over $k[[\sqrt{t}]]$. A computation shows that $\mathrm{Cl}_0(\mathcal{Q}).f$ has rank 2 over $k[[\sqrt{t}]]$ and that ideals $\mathrm{Cl}_0(\mathcal{Q}).f_{\sqrt{t}}$ and $\mathrm{Cl}_0(\mathcal{Q}).f_{-\sqrt{t}}$ merge into a rank 2 ideal at $t = 0$. $\square$

The orthogonal Grassmanian of $\mathcal{Q}$ consists of two disjoint conics away from the degenerate locus. These merge into the same conic over the degenerate locus. It follows that $BS(B_x) \simeq OGr(2, 4, \mathcal{Q}_x)$ which is a conic bundle on M.

We need one more result to proceed. Given a curve S of genus 2, there is a pencil of quadrics $\mathcal{X}$ in $\mathbb{P}^5$ such that S is isomorphic to the Stein factorization of the family of planes in $\mathcal{X}$. The base locus U of $\mathcal{X}$ is a complete intersection of two quadrics. Given any point $u \in U$, and a quadric Q in $\mathcal{X}$, $T_u(Q) \cap Q$is a cone over a quadric surface. So projecting away from u we get a quadric surface bundle on S. The variety of lines in this quadric surface bundle is a conic bundle P on S. Varying u we get a conic bundle $\mathcal{P}$, on $U \times S$ which corresponds to an element in the Brauer group $\mathrm{Br}(U \times S) = H^2(U \times S, \mathcal{O}^*)$. Since the Brauer group of a curve is trivial, this conic bundle lifts to a rank 2 vector bundle $\mathcal{F}$ on $U \times S$. New stead proves that U is a moduli space of rank 2 vector bundles on S with odd degree and that $\mathcal{F}$ is a universal family [20]. Equivalently, this means that given two distinct points p and q in U, the corresponding conic bundles are not isomorphic.

Now we prove our first main theorem.

Theorem 4.6 *Let $\phi : M \to \mathbb{P}^2$ be a K3 surface realised as a double cover of $\mathbb{P}^2$ branched along a smooth sextic C. Let L be an even theta characteristic on C such that $H^0(L) = 0$ and $\bar{\mathcal{Q}}$ the net of quadrics in $\mathbb{P}^5$ obtained from a minimal resolution of L. Let X be the base locus of $\bar{\mathcal{Q}}$. Then X is a moduli space of rank 4 Azumaya algebras on M with $\chi = 0$, and $c_2 = 8$.*

Proof. For each $x \in X$, we construct a rank 4 Azumaya algebra B_x with $\chi(B_x) = 0$ and $c_2(B_x) = 8$, as in Theorem 4.4. Let $\mathcal{M}$ denote the moduli stack of rank 4 Azumaya algebras on M with $c_2 = 8$ and same underlying gerbe as B_x for some $x \in X$. Since $h^0(B_x) = 1$, it follows that $h^1(B_x) = 2$. From the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{O} \to B_x \to B_x/\mathcal{O} \to 1$$

we get $h^1(B_x/\mathcal{O}) = 2$ and $h^2(B_x/\mathcal{O}) = 0$. Therefore by Proposition 3.8 $\mathcal{M}$ is two dimensional.

Recall that an Azumaya algebra A is simple if $h^0(A) = 1$. Given a simple Azumaya algebra A of rank r^2(equivalently degree r) on a K3 surface we know that $c_2(A) \geq 2r^2 - 2$. So for simple rank 4 Azumaya algebras $c_2 \geq 6$.

By Corollary 6.6 it follows that the Azumaya algebras B_x have minimal c_2, so $\mathcal{M}$ is a smooth projective surface (see Theorem 2.10).

The underlying gerbe of B_x is an element of the discrete group $H^2(M, \mu_2)$, hence stays constant as we vary x in X. So we have a morphism $\psi : X \to \mathcal{M}$. We need to show that it is injective to conclude that $X \simeq \mathcal{M}$

The Brauer-Severi variety of B_x, $BS(B_x)$, is isomorphic to the orthogonal Grassmanian of $\mathcal{Q}_x$, and is a conic bundle on M. Let S be a smooth genus 2 curve in M, which is an element of the linear system $\mathcal{O}_M(1)$. Then S corresponds to a pencil of quadrics $\mathcal{X}$ contained in $\mathcal{Q}$. The base locus U of this pencil is a moduli space of rank 2 vector bundles of odd degree, on S [20]. Each point $x \in U$ defines a conic bundle P_x on S which lifts to a rank 2 vector bundle. When $x \in X \subset U$, $P_x \simeq OGr(2, 4, \mathcal{Q}_{x_{|S}})$ and by so Theorem 4.5 is isomorphic to $BS(B_x)_{|S}$.

Suppose there are two distinct points x and y in X such that the Brauer-Severi varieties $BS(B_x)$ and $BS(B_y)$ are isomorphic. Then $BS(B_x)_{|S} \simeq BS(B_y)_{|S}$ are isomorphic as conic bundles, and hence lift to isomorphic rank 2 vector bundles on S, which is a contradiction. Therefore the map $\psi : X \to \mathcal{M}$ is injective and $X \simeq \mathcal{M}$.

$\square$

In fact there is another construction which gives a universal family of Azumaya algebras on $X \times M$.

Consider the incidence correspondence in $\mathbb{P}^2 \times \mathbb{P}^5 \times Gr(2, 5)$ given by

$$I = \{(a, x, \Lambda) | a \in \mathbb{P}^2, x \in X, \Lambda \subset T_x(\overline{Q}_a) \cap \overline{Q}_a\}$$

This is a conic bundle $\pi : I \to X \times M$ such that for each $x \in X$, $I_{|x \times M} \simeq OGr(2, 4, \mathcal{Q}_x) \simeq BS(B_x)$. Consider the short exact sequence as in [6]

$$0 \to \mathcal{O} \to J \to T_{I/X \times M} \to 0,$$

where $T_{I/X \times M}$ is the relative tangent sheaf. Then $\mathcal{B} = \pi_*(\mathcal{E}nd\ J) \simeq \pi_* J$ is an Azumaya algebra on $X \times M$ and $BS(\mathcal{B}) \simeq I$. So in fact $\mathcal{B}$ is a universal family on $X \times M$ such that $\mathcal{B}_{|x \times M} \simeq B_x$. Moreover $\mathcal{B}_{|X \times m} \simeq F \otimes F^*$ where F is a spinor bundle on X corresponding to $m \in M$. The variety $BS(B_x) \simeq I_{|x \times M}$ is a conic threefold. We use the results from [6] to compute its Chern invariants.

$$\begin{aligned}\chi_{top}(BS(B_x)) &= 2\chi_{top}(M) = 48,\\ \chi(\mathcal{O}_{BS(B_x)}) &= \chi(\mathcal{O}_M) = 2,\\ K^3_{BS(B_x)} &= c_2(B_x) = 8\end{aligned}$$

In general M is a non fine moduli space of sheaves on X because the conic bundle I does not lift to a vector bundle. In some special cases M is a fine moduli space and the moduli problem is actually symmetric [12]. Theorem 4.6 is like an inverse of this moduli problem. However there is a subtlety here because we have to make a choice of a theta characteristic. So whereas given an X in $\mathbb{P}^5$ there is a unique M, given an M there isn't a unique corresponding K3 surface X. There are $2^9(2^{10}+1)$ even theta characteristics so there are $2^9(2^{10}+1)$ choices for X all of which are non isomorphic as K3 surfaces. The obstruction to the existence of a universal sheaf is an element $\alpha \in Br(M)$ which can be described in terms of a Hodge isometric embedding $T_X \to T_M$ via the Fourier-Mukai map on cohomology. There are $2^9(2^{10}+1)$ even sublattices of T_M, of index 2, which arise as transcendental lattices of a K3 surface X [23]. So given a K3 surface M we get $2^9(2^{10}+1)$ non isomorphic division algebras on M whose moduli spaces are the corresponding distinct K3 surfaces X.

In fact by extending some results of Bridgeland we prove a derived equivalence $D(B) \simeq D(X)$, where $D(B)$ is the category of modules over Azumaya algebras B on M with the same underlying gerbe as B_x, see Theorem 5.8.

An equivalent formulation of the inverse of the Mukai correspondence is in terms of twisted sheaves, so X can be viewed as a moduli space of twisted sheaves on M [7], [11] [24].

5 Derived Equivalence

Let X be a K3 surface. For the product of $X \times M$ we let π_M, π_X be the projections, and we denote the fibres over points $m \in M$ and $x \in X$ by X_m, M_x. Let v be a Mukai vector such that the moduli space M of vector bundles with Mukai vector v is a surface. The following results due to Mukai are well known [18], [19].

Proposition 5.1 *The moduli space M of stable vector bundles E_m with Mukai vector v, if non empty and compact, is a K3 surface. There is a*

quasi-universal sheaf $\mathcal{E}$ on $X \times M$ such that for each point m of M we have $\mathcal{E} \otimes \mathcal{O}_{X_m} \simeq E_m^n$.

From the quasi-universal sheaf $\mathcal{E}$ we construct an Azumaya algebra on M.

Proposition 5.2 *Let $B = \pi_{M*} \mathcal{E}nd_{X \times M} \mathcal{E}$. Then B is an Azumaya algebra on M and $\pi_{M*}\mathcal{E}$ is a B-module and $\mathcal{E}$ is a $\pi_M{}^*B$ module.*

Proof. Since each E_m is a simple, we know that $\mathrm{End}(E_m) = k$, and so

$$H^0(X_m, \mathcal{E}nd\,(\mathcal{E}) \otimes \mathcal{O}_{X_m}) \simeq H^0(X_m, \mathcal{E} \otimes \mathcal{E}^* \otimes \mathcal{O}_{X_m})$$

$$\simeq H^0(X, (E_m^n) \otimes (E_m)^{n*}) \simeq \mathrm{End}(E_m^n, E_m^n) \simeq k^{n \times n}$$

and since B is a coherent sheaf of $\mathcal{O}_X$ algebras, by cohomology and base change we see that B is Azumaya. $\square$

Let A be an $\mathcal{O}_X$-algebra and let B be an $\mathcal{O}_M$-algebra. By this we mean that $\mathcal{O}_X$ is central in A. Let B^o be the opposite algebra of B. As usual we define $A \boxtimes B^o = \pi_X^* A \otimes \pi_M^* B^o$. Following Van den Bergh, we define an $A - B$ bimodule to be a quasi-coherent sheaf $\mathcal{E}$ on $X \times M$ that has the structure of an $A \boxtimes B^o$ module. We sometimes write ${}_A\mathcal{E}_B$ to emphasise the bimodule structure. We define functors

$$\mathcal{E} \otimes_B - = \pi_{X*}(\mathcal{E} \otimes_{\pi_M^* B^o} \pi_M^* -)$$

from the category of left B-modules to right A-modules and

$$- \otimes_A \mathcal{E} = \pi_{M*}(\pi_X^* - \otimes_{\pi_X^* A} \mathcal{E})$$

from the category of right A-modules to left B-modules.

We use the methods of Bridgeland [4] to determine conditions for these functors to induce a derived equivalence. Let X and M be K3 surfaces, $A = \mathcal{O}_X$ and $B = \pi_{M*} \mathcal{E}nd\ \mathcal{E}$ as above.

The main fact needed to generalise Bridgeland proof is the following easy lemma.

Lemma 5.3 *There is a natural isomorphism*

$$k^{1 \times n} \otimes_{k^{n \times n}} k^{n \times 1} \simeq k.$$

Proof. We can verify this directly or via the Yoneda lemma and the adjunction of tensor and Hom. $\square$

For every point m of M the fibre of B is a matrix algebra $B_m \simeq k^{n\times n}$. There is a simple B-module $V_m \simeq k^{n\times 1}$ supported at m that is unique up to isomorphism. Note that the modules V_m are not the fibres of vector bundle in any way that is consistent with their structures of a B-module unless B is trivial in the Brauer Group $\operatorname{Br} M$. Let $\Omega = \{V_m | m \in M\}$. Following [4] we call Ω a spanning set for $D(B)$ if for all i, m we have either of the conditions

$$\operatorname{Hom}_{D(B)}(V_m, C[i]) = 0$$

$$\operatorname{Hom}_{D(B)}(C[i], V_m) = 0$$

then $C \simeq 0$. Since every coherent B-module has a non-trivial surjection to some simple module V_m, we have the following lemma.

Lemma 5.4 *The modules Ω form a spanning set for the derived category $D(B)$.*

Proof. First note that since $\omega_B \simeq B$ the two conditions are equivalent. So we focus our attention on the latter. We know that for any coherent B-module there is some m so that $\operatorname{Hom}_B(F, V_m) \neq 0$, since F must have a simple quotient. So if $\operatorname{Hom}_B(F, V_m) = 0$ for all m then $F = 0$. Now let C be an object of $D(B)$ such that $C \neq 0$. Then there is a spectral sequence

$$\operatorname{Ext}^p_B(\mathcal{H}^q(C), V_m) \Rightarrow \operatorname{Hom}_{D(B)}(C[q-p], V_m).$$

Since we are in the bounded derived category there is a maximal q such that $\mathcal{H}^q(C) \neq 0$, so for some $m \in M$ there is a homomorphism in $\operatorname{Hom}_B(H^q(C), V_m) \neq 0$. Since we have $p = 0$ and q is maximal we have that $q - p$ is maximal, and so $\operatorname{Hom}_B(H^q(C), V_m)$ can not vanish in the spectral sequence. $\square$

We need another lemma concerning an adjunction that we prove using the Serre functor. Recall that if we let $f^* : \mathcal{O}_M \to B$ be the structure map then the dualizing bimodule for B is

$$\omega_B := f^!\omega_X := \operatorname{Hom}_B(B, \omega_X) = B^\vee \otimes \omega_X$$

which satisfies Serre duality

$$\operatorname{Ext}^i_B(a, b)^* = \operatorname{Ext}^{n-i}_B(b, \omega_B \otimes a).$$

In particular $S_B := \omega_B[n] \otimes_B -$ defines the Serre functor on $D(B)$ so we have the equation

$$\mathrm{Hom}_{D(B)}(a,b)^* = \mathrm{Hom}_{D(B)}(b, S(a)).$$

In our particular case where M is K3 and B is Azumaya, we have that $\omega_B \simeq B$ and $\omega_X \simeq \mathcal{O}_X$ as bimodules.

Lemma 5.5 *For an $\pi_M^* B$-module a and a B-module b, we have the adjunction*

$$\mathrm{Hom}_{D(B)}(R\pi_{M*}a, b) = \mathrm{Hom}_{D(\pi_M^*}(a, \omega_X[n] \otimes \pi_M^* b).$$

Proof. We start with the left hand side and apply the Serre functor to derive that

$$\begin{aligned}
&\mathrm{Hom}_{D(B)}(R\pi_{M*}a, b) \simeq \mathrm{Hom}_{D(B)}(b, \omega_B[n] \otimes_B R\pi_{M*}a)^* \\
&\simeq \mathrm{Hom}_{D(B)}(\omega_B^\vee[-n] \otimes_B b, R\pi_{M*}a)^* \simeq \mathrm{Hom}_{D(\pi_M^* B)}(\pi_M^*(\omega_B^\vee[-n] \otimes_B b), a)^* \\
&\simeq \mathrm{Hom}_{D(\pi_M^* B)}(\pi_M^* \omega_B^\vee[-n] \otimes \pi_M^* b, a)^* \\
&\simeq \mathrm{Hom}_{D(\pi^* MB)}(a, \omega_X[n] \boxtimes \omega_B[n] \otimes (\pi_M^* \omega_B^\vee[-n] \otimes \pi_M^* b) \\
&\simeq \mathrm{Hom}_{D(\pi^* MB)}(a, \omega_X[n] \otimes \pi_M^* b).
\end{aligned}$$

□

Let $F : D(B) \to D(X)$ be the derived functor of $\mathcal{E} \otimes_B -$. Note that despite the notation, this is a right derived functor. Since $\mathcal{E}$ is a flat B-module we only need to derive the π_{X*}. So

$$DF(-) = R\pi_{X*}(\mathcal{E} \otimes \pi^* B^o -).$$

By results of Bridgeland, we only need verify that the following conditions are satisfied.

Theorem 5.6 *The derived functor of F induces an equivalence of derived categories if*

1.
$$\mathrm{Hom}_X(FV_m, FV_m) = k.$$

2.
$$\mathrm{Ext}^i_X(E_{m_1}, E_{m_2}) = 0 \textit{ for all } i \textit{ and } m_1 \neq m_2$$

.

3.

$$E_m \otimes \omega_X \simeq E_m.$$

Proof. We calculate

$$FV_m = R\pi_{X*}(\mathcal{E} \otimes_{\pi_M^* B} \pi_M^* V_m)$$

$$\simeq R\pi_{X*}(E_m^n \otimes_{\mathcal{O}_X^{n\times n}} \pi_M^* k^{n\times 1})$$

$$\simeq R\pi_{X*} E_m \simeq E_m.$$

Since each E_m is simple it follows that $\mathrm{Hom}_X(E_m, E_m) = k$. Since M is a moduli space of sheaves on X, (2) is satisfied. Condition (3) is a consequence of the fact that X is a K3 surface and hence $\omega_X \simeq \mathcal{O}_X$.

□

Lemma 5.7 *Let*

$$FL = R(\mathrm{Hom}(\pi_M^* B, \mathcal{E}) \otimes \pi_X^* \omega_X[\dim X] \otimes_X -).$$

and

$$FR = R(\mathrm{Hom}(\pi_M^* B, \mathcal{E}) \otimes \pi_M^* \omega_B[\dim M] \otimes_X -).$$

then FR and FL are left and right adjoints to $DF = R(\mathcal{E} \otimes_B -)$.

So by a straightforward adaptation of the methods in [4] we have the following theorem.

Theorem 5.8 *Let M be a K3 surface which is a moduli space of stable sheaves on a K3 surface X, with Mukai vector v. Let $\mathcal{E}$ be a quasi-universal sheaf on $X \times M$. Let $B = \pi_{M*}(\mathcal{E}nd\ \mathcal{E})$. Let $D(B)$ denote the derived category of left $B-$modules on M. Then there is a derived equivalence $D(B) \simeq D(X)$.*

We apply this to our earlier example where X is a degree 8 K3 surface in $\mathbb{P}^5$ given as the base locus of a net of quadrics $\mathcal{Q}$ and M a double cover of $\mathbb{P}^2$ branched over $V(\det(\mathcal{Q})$. A quasi universal sheaf $\mathcal{E}$ exist for this problem such that $\mathcal{E}_{|X_m} \simeq E_m^2$ [15]. Then $B := \pi_{M*}(\mathcal{E}nd\ \mathcal{E})$ is a rank 4 Azumaya algebra on M, and the functor $DF(-) = R\pi_{X*}(\mathcal{E} \otimes \pi^* B^o -)$ induces a derived equivalence $D(B) \simeq D(X)$.

For an equivalent formulation in terms of twisted sheaves see [7], [11].

6 Moduli Spaces

Let A be a simple Azumaya algebra of rank r^2 over X. Consider M, the moduli stack of Azumaya algebras that are Morita equivalent to A. Let M_A be a component of M that includes A. The gerbe underlying A in $H^2(X, \mu_r)$ is constant over M_A since this group is discrete.

In this section we show that if A and A' are Azumaya algebras of degree r that have the same underlying gerbe, then $2r \mid c_2(A) - c_2(A')$. This shows that if the second Chern class jumps down in M_A then it must jump down by at least $2r$. Hence if A is simple and has its second Chern class within $2r$ of a lower bound, then it has minimal c_2 and by Theorem 2.10, M_A is a proper moduli space.

We need the following diagrams:

Lemma 6.1 *There is a commutative diagram*

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, equal] & \mathbb{G}_m & \\
\mathbb{G}_m \arrow[u] \arrow[r] & \mathrm{GL}_r \arrow[u] \arrow[r] & \mathrm{PGL}_r \\
\mu_r \arrow[u] \arrow[r] & \mathrm{SL}_r \arrow[u] \arrow[r] & \mathrm{PSL}_r \arrow[u]
\end{tikzcd}$$

of short exact sequences. There is also a commutative diagram

$$\begin{tikzcd}
\mu_r \arrow[r, equal] \arrow[d] & \mu_r \arrow[r, equal] \arrow[d] & \mu_r \arrow[d] \\
\mathrm{SL}_r \arrow[d] & G \arrow[l] \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[d, "(-)^r"] \\
\mathrm{PGL}_r & \mathrm{GL}_r \arrow[l] \arrow[r, "\det^{-1}"] & \mathbb{G}_m
\end{tikzcd}$$

where the bottom squares are both Cartesian.

The map $(-)^r : \mathbb{G}_m \to \mathbb{G}_m$ is the r^{th} power map. The proof is a straight forward diagram chase. So the group G above can defined as either of

$$G = \mathrm{GL} \times_{\mathrm{PGL}} \mathrm{SL} \simeq \mathrm{GL} \times_{\mathbb{G}_m} \mathbb{G}_m.$$

So we have a commutative diagram

$$\begin{array}{ccc} H^1(\mathrm{PSL}_r) & \longrightarrow & H^2(\mu_r) \\ \| & & \downarrow \\ H^1(\mathrm{PGL}_r) & \longrightarrow & H^2(\mathbb{G}_m) \end{array}$$

Given an Azumaya algebra we obtain a class in $H^1(M, \mathrm{PGL}_r)$ which we can map to an element of $H^2(\mu_r)$. We call this the underlying gerbe of the Azumaya algebra. The image in $H^2(M, \mathbb{G}_m)$ is the corresponding element in the Brauer group.

The previous lemma shows that given a curve C we have a commutative diagram.

$$\begin{array}{ccc} H^1(C, GL_r) & \xrightarrow{\det^{-1}} & \mathrm{Pic}\, C \\ \downarrow & & \downarrow \\ H^1(C, PGL_r) & \longrightarrow & H^2(C, \mu_r) \end{array}$$

So if we have a vector bundle V on C with rank r and degree v then the value of $-v \pmod r$ is an invariant of $V \otimes V^*$ and can be considered to be the image of the composition

$$H^1(C, PGL_r) \xrightarrow{\simeq} H^1(C, PSL_r) \to H^2(C, \mu_r).$$

To proceed we need several results from [1].

Definition 6.2 Let A be an Azumaya algebra on a smooth surface X. Let Y be a smooth curve in X, the restriction $A \otimes \mathcal{O}_Y$ splits by Tsen's Theorem so there is a vector bundle V on Y such that $A \otimes \mathcal{O}_Y \simeq \mathcal{E}nd\,(V) \simeq V \otimes V^*$. Let $F \subset V$ be a subsheaf of V which is locally a direct summand. There are canonical surjective maps of right $\mathcal{A}$-modules:

$$A \to A_Y \simeq V^* \otimes V \to F^* \otimes V.$$

Let K_A be the kernel of the composed map, so we have an exact sequence of right A-modules.

$$0 \to K \to A \to F^* \otimes V \to 0.$$

Artin and deJong, [1] 8.2, define $A' = A'(A, Y, V, F) := \mathcal{E}nd\ K_A$ to be the *elementary transformation* of A.

Proposition 6.3 *[1] 8.2.11 Let A, A' be two Azumaya algebras in the same central simple algebra $k(A)$ over a smooth projective surface. Then there is a smooth curve $Y \subset X$ and a local direct summand $F \subset V$ such that A' is the elementary transformation $A' = A'(A, Y, V, F)$. Furthermore F can be taken to be an invertible sheaf. Furthermore we may also replace Y by a smooth curve linearly equivalent to a a multiple of Y.*

Let $Q = V/F$, be the quotient bundle on Y, and let us denote the ranks and degrees of V, F, Q as $v_0, v_1, f_0, f_1, q_0, q_1$, respectively. Note that $v_0 = f_0 + q_0 = r$ and $v_1 = f_1 + q_1$. We need the following formula:

Proposition 6.4 *[1] 8.3.1 Let $A' = A'(A, Y, V, F)$ be an elementary transformation of A, then*

$$c_2(A') - c_2(A) = -f_0 q_0 C^2 + 2(f_0 q_1 - f_1 q_0)$$

Note: The formula as stated in the manuscript [1] has a minor error.

Proposition 6.5 *Let A and A' be Morita equivalent Azumaya algebras of degree r with the same underlying gerbe in $H^2(X, \mu_r)$. Then $2r \mid c_2(\mathcal{A}) - c_2(\mathcal{A}')$.*

Proof. By Proposition 7.3 [1], A' is the elementary transformation of A over a smooth curve C in M. This implies that there is the data of a decomposition $A|_C = V \otimes V^*$ and a subvector bundle F of V of rank one, from which one can build A'. We let $Q = V/F$ and let $q_0, q_1, f_0, f_1, r, v_1$ be ranks and degrees of Q, F and V respectively. So $q_0 + f_0 = r$ and $q_1 + f_1 = v_1$. We have that $f_0 = 1$ and $q_0 = r - 1$. Note that the class of C in $H^2(X, \mu_r)$ is the difference of the classes of A, A' in $H^2(X, \mu_r)$ by [1] Corollary 8.2.10. We can also choose C sufficiently large, as long its Chern class is zero in $H^2(X, \mu_r)$. So we choose $C = rD$ such that $2r \mid C^2$. Then (see Proposition 6.4)

$$\begin{aligned} & c_2(A') - c_2(A) \\ &= -f_0 q_0 C^2 + 2(f_0 q_1 - f_1 q_0) \\ &= -f_0 q_0 C^2 + 2(v_1 - f_1 r) \\ &= -(r-1)C^2 + 2(v_1 - f_1 r). \end{aligned}$$

So if we can show that $r \mid v_1$ then we are done.

The value of $-v_1 \pmod r$ is an invariant of $V \otimes V^*$ over C and is the image of the composition

$$H^1(C, PGL_r) \overset{\cong}{\to} H^1(C, PSL_r) \to H^2(C, \mu_r).$$

Recall that our Azumaya algebra A gives us an element of

$$H^1(X, PGL_r) \overset{\cong}{\to} H^1(X, PSL_r) \to H^2(X, \mu_r) \to H^2(C, \mu_r).$$

The image of A in $H^2(C, \mu_r)$ via the above mapping $H^2(X, \mu_r) \to H^2(C, \mu_r)$, takes the value $-v_1 \pmod r$.

Now since $C = rD$, we conclude that the image of $H^2(X, \mu_r) \to H^2(C, \mu_r)$ is 0. So we must have that $r \mid v_1$. □

Corollary 6.6 *Let A be an Azumaya algebra of degree r and suppose that there is a lower bound $c_2(A') \geq k$ for all Morita equivalent Azumaya algebras A' of degree r. If $c_2(A) < k + 2r$ then A has minimal c_2.*

Proof. Let A' be an Azumaya algebra of degree r, Morita equivalent to A. Suppose there exists an A' such that $c_2(A') < c_2(A)$. Then by the above Proposition $2r \mid c_2(A) - c_2(A')$ which implies that $c_2(A) \geq k + 2r$ which is a contradiction. Therefore A must have minimal c_2.

□

References

[1] M. Artin, J. de Jong, *Stable orders over surfaces*, Manuscript 2004. `http://www.math.lsa.umich.edu/courses/711/ordersms-num.pdf`

[2] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*. Springer-Verlag, Berlin, 1984. x+304 pp. ISBN: 3-540-12172-2

[3] W. Bichsel and M. Knus, *Quadratic forms with values in line bundles*, Recent advances in real algebraic geometry and quadratic forms (Berkeley, CA, 1990/1991; San Francisco, CA, 1991), 293306, Contemp. Math., 155, Amer. Math. Soc., Providence, RI, 1994.

[4] T. Bridgeland, *Equivalences of triangulated categories and Fourier-Mukai transforms*. International Congress of Mathematicians. Vol. II, 563582, Eur. Math. Soc., Zrich, 2006.

[5] R. Buchweitz, D. Eisenbud and J. Herzog) *Cohen-Macaulay Modules on Quadrics*, Singularities, Representation of Algebras, and Vector Bundles , Proceedings, Lambrecht 1985, Springer Lecture Notes 1273, 58-95, (1987).

[6] D. Chan and C. Ingalls, *Conic bundles and Clifford algebras*, preprint arXiv:1101.1705v1

[7] A. Caldararu, *Non-fine moduli spaces of sheaves on K3 surfaces*, Internat. Math. Res. Notices (2002), no. 20, 1027-1056.

[8] P. Gille and T. Szamuely, "Central Simple Algebras and Galois Cohomology," Cambridge Studies in Adv. Math. (101), Cambridge Univ. Press, (2006).

[9] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley and Sons], New York, 1978. xii+813 pp. ISBN: 0-471-32792-1

[10] D. Huybrechts and S. Schrer, *The Brauer group of analytic K3 surfaces.* Int. Math. Res. Not. 2003, no. 50, 26872698.

[11] , D. Huybrechts, P. Stellari, *Proof of Caldararu's conjecture,* Moduli spaces and arithmetic geometry, 3142, Adv. Stud. Pure Math., 45, Math. Soc. Japan, Tokyo, 2006.

[12] C. Ingalls and M. Khalid, *Rank two sheaves on K3 surfaces: A special construction*, preprint available at http://arxiv.org/abs/0803.0179

[13] M. Kapranov, *On the derived category and K-functor of coherent sheaves on intersections of quadrics*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 52 (1988), no. 1, 186–199; translation in Math. USSR-Izv. 32 (1989), no. 1, 191204

[14] M. Knus, *Quadratic forms, Clifford Algebras, and spinor bundles*, course lecture notes.

[15] M. Khalid, *On K3 correspondences*. J. Reine Angew. Math. 589 (2005), 5778.

[16] Y. Manin, *Topics in noncommutative geometry*, M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1991. viii+164 pp. ISBN: 0-691-08588-9

[17] J. Milne, *Étale cohomology*, Princeton Mathematical Series, 33. Princeton University Press, Princeton, N.J., 1980. xiii+323 pp. ISBN: 0-691-08238-3

[18] S. Mukai, *On the moduli space of bundles on K3 surfaces I, Vector bundles on algebraic varieties*, Bombay 1984, Tata Inst. Fund. Res. Stud. Math. 11 (1987), 314-413.

[19] S. Mukai, *Moduli of vector bundles on K3 surfaces and symplectic manifolds*, Sugako Expos. 1 (1988), no.2, 139-174, Sugako 39 (1987) 216-235.

[20] P. Newstead, *Stable bundles of rank* 2*2 and odd degree over a curve of genus 2*. Topology 7 1968 205215.

[21] T. Pantev and E. Sharpe, *GLSMs for gerbes (and other toric stacks)*. Adv. Theor. Math. Phys. 10 (2006), no. 1, 77121.

[22] A. Tjurin, *The intersection of quadrics* (Russian), Uspehi Mat. Nauk. 30 (1975) no.6 (186), 51-99.

[23] B. van Geemen, *Some remarks on Brauer groups of K3 surfaces*, Advances in Mathematics, **197** (2005), 222–247.

[24] K. Yoshioka, *Moduli spaces of twisted sheaves on a projective variety. Moduli spaces and arithmetic geometry*, 1–30, Adv. Stud. Pure Math., 45, Math. Soc. Japan, Tokyo, 2006.